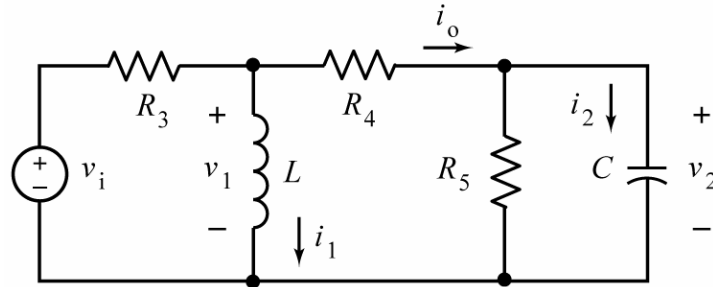


State Equations for Dynamic Circuits

Consider a dynamic circuit that does not contain any

- Loops consisting entirely of capacitors and voltage sources.
- Cutsets consisting entirely of inductors and current sources.

For example, consider



The input to this circuit is the source voltage, v_i . The output is the current i_o . The inductor current, i_1 , and capacitor voltage, v_2 , are called state variables. We can represent this circuit by differential equations of the form

$$\begin{aligned} \frac{d i_1}{d t} &= a_{11} i_1 + a_{12} v_2 + b_1 v_i \\ \frac{d v_2}{d t} &= a_{21} i_1 + a_{22} v_2 + b_2 v_i \end{aligned}$$

Then the output can be expressed as

$$i_o = c_1 i_1 + c_2 v_2 + d v_i$$

The coefficients, a_{11} , a_{12} , a_{21} , a_{22} , b_1 , b_2 , c_1 , c_2 and d are constants that depend on the values of L , C , R_3 , R_4 and R_5 . These equations are called state equations. We can write the state equations in matrix form:

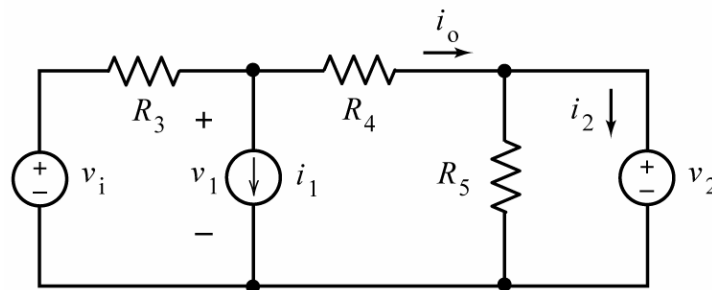
$$\begin{bmatrix} \frac{d i_1}{d t} \\ \frac{d v_2}{d t} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} v_i$$

$$i_o = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} i_1 \\ v_2 \end{bmatrix} + d v_i$$

This looks pretty complicated. We might ask why we would want to do such a thing. We will see that

- Writing the state equations isn't particularly difficult.
- MATLAB provides differential equation solvers that we can use to plot i_1 , v_2 and i_o .

Let's write the state equations. With the benefit of hind sight, we will replace the inductor by a current source and replace the capacitor by a voltage source to get



(The orientations of the new sources are chosen to agree with the orientations of i_1 and v_2 .) This circuit is a static circuit since it does not contain capacitors or inductors. We could analyze this circuit by writing node equations or mesh equation, but it seems easier to use superposition. The objective of the analysis is the express

v_1 , the voltage across the current source that replaced the inductor,

i_2 , the current is the voltage source that replaced the capacitor,

and

i_o , the output of the original circuit,

as functions of the input to this circuit, v_i , and the state variables, i_1 , and, v_2 .

Using superposition we find

$$v_1 = -\left(\frac{R_3 R_4}{R_3 + R_4}\right) i_1 + \left(\frac{R_3}{R_3 + R_4}\right) v_2 + \left(\frac{R_4}{R_3 + R_4}\right) v_i$$

$$i_2 = -\left(\frac{R_3}{R_3 + R_4}\right) i_1 - \left(\frac{1}{R_5} + \frac{1}{R_3 + R_4}\right) v_2 + \left(\frac{1}{R_3 + R_4}\right) v_i$$

$$i_o = -\left(\frac{R_3}{R_3 + R_4}\right) i_1 - \left(\frac{1}{R_3 + R_4}\right) v_2 + \left(\frac{1}{R_3 + R_4}\right) v_i$$

These equations also represent the original circuit. In the original circuit, it is also true that

$$v_1 = L \frac{di_1}{dt} \quad \text{and} \quad i_2 = C \frac{dv_2}{dt}$$

Combining these equations gives

$$L \frac{di_1}{dt} = - \left(\frac{R_3 R_4}{R_3 + R_4} \right) i_1 + \left(\frac{R_3}{R_3 + R_4} \right) v_2 + \left(\frac{R_4}{R_3 + R_4} \right) v_i$$

$$C \frac{dv_2}{dt} = - \left(\frac{R_3}{R_3 + R_4} \right) i_1 - \left(\frac{1}{R_5} + \frac{1}{R_3 + R_4} \right) v_2 + \left(\frac{1}{R_3 + R_4} \right) v_i$$

$$i_o = - \left(\frac{R_3}{R_3 + R_4} \right) i_1 - \left(\frac{1}{R_3 + R_4} \right) v_2 + \left(\frac{1}{R_3 + R_4} \right) v_i$$

Dividing both sides of the first equation by L and dividing both sides of the first equation by C we have the state equations:

$$\frac{di_1}{dt} = - \left(\frac{R_3 R_4}{L(R_3 + R_4)} \right) i_1 + \left(\frac{R_3}{L(R_3 + R_4)} \right) v_2 + \left(\frac{R_4}{L(R_3 + R_4)} \right) v_i$$

$$\frac{dv_2}{dt} = - \left(\frac{R_3}{C(R_3 + R_4)} \right) i_1 - \left(\frac{1}{C R_5} + \frac{1}{C(R_3 + R_4)} \right) v_2 + \left(\frac{1}{C(R_3 + R_4)} \right) v_i$$

$$i_o = - \left(\frac{R_3}{R_3 + R_4} \right) i_1 - \left(\frac{1}{R_3 + R_4} \right) v_2 + \left(\frac{1}{R_3 + R_4} \right) v_i$$

Here's a MATAB script based on this analysis:

```
tspan=[0,1]; % solution time interval
xo=[1; 0]; % xo must be a column

% description of the circuit in example 1

Vi = 0;

L = 4; % H
C = 0.125; % F
R3 = 20; % Ohms
R4 = 20;
R5 = 10;

a11 = -R3*R4/L/(R3+R4);
a12 = R3/L/(R3+R4);
a21 = -R3/C/(R3+R4);
a22 = -1/R5 - 1/C/(R3+R4);
b1 = R4/L/(R3+R4);
b2 = 1/C/(R3+R4);
c1 = -R3/(R3+R4);
c2 = -1/(R3+R4);
d = 1/(R3+R4);

A = [a11 a12; a21 a22];
B = [b1; b2];

[t,x]=ode45(@steqn, tspan, xo, [], A, B, Vi);

io = [c1 c2]*x' + d*Vi;
plot(t, io')
```

The above script uses this function:

```
function xdot=steqn(t,x,A,B,Vi)

xdot = A*x + B*Vi;
```

Here's the plot produced by the MATLAB script:

